PHYSICAL AND ECONOMIC OPTIMUM FOR SINGLE INPUT

Let y = f(x) be a response function. Here x stands for the input that is kgs of fertilizer applied per hectare and y the corresponding output that is kgs of yield per hectare.

We know that the maximum is only when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

This optimum is called physical optimum. We are not considering the profit with respect to the investment, we are interested only in maximizing the profit.

Economic optimum

The optimum which takes into consideration the amount invested and returns is called the economic optimum.

$$\frac{dy}{dx} = \frac{P_x}{P_y}$$

where $P_x \rightarrow$ stands for the per unit price of input that is price of fertilizer per kgs.

$P_y \rightarrow$ stands for the per unit price of output that is price of yield per kgs.

Problem

The response function of paddy is y = 1400 + 14.34x -0.05 x^2 where x represents kgs of nitrogen/hectare and y represents yield in kgs/hectare. 1 kg of paddy is Rs. 2 and 1 kg of nitrogen is Rs. 5. Find the physical and economic optimum. Also find the corresponding yield.

Solution

y = 1400 + 14.34x -0.05 x^2

$$\frac{dy}{dx} = 14.34 - 0.1x$$

$\frac{d^2y}{dx^2} = -0.1 =$ negative value

ie. $\frac{d^2y}{dx^2} < 0$

Therefore the given function has a maximum point.

Physical Optimum

$$\frac{dy}{dx} = 0$$

i.e 14.34-0.1x = 0

-0.1 x = -14.34

x = $\frac{14.34}{0.1} = 143.4$ kgs/hectare

therefore the physical optimum level of nitrogen is 143.4 kgs/hectare.

Therefore the maximum yield is

Y = 1400 + 14.34(143.4) -0.05(143.4)^2

= 2428.178 kgs/ hectare.

Economic optimum

$$\frac{dy}{dx} = \frac{P_x}{P_y}$$

Given

Price of nitrogen per kg = P_x = 5

Price of yield per kg = P_y = 2

Therefore $\frac{dy}{dx}$ = 14.34-0.1x = $\frac{5}{2}$

28.68 - 0.2 x = 5

- 0.2 x = 5 - 28.68

x = $\frac{23.68}{0.2} = 118.4$ kgs/hectare

therefore the economic optimum level of nitrogen is 118.4 kgs/hectare.

Therefore the maximum yield is

Y = 1400 + 14.34(118.4) -0.05(118.4)^2

= 2396.928 kgs/ hectare.

Maxima and Minima of several variables with constraints and without constraints

Consider the function of several variables

$$y = f(x_1, x_2 \ldots\ldots\ldots x_n)$$

where $x_1, x_2 \ldots\ldots\ldots\ldots x_n$ are n independent variables and y is the dependent variable.

Working Rule

Step 1: Find all the first order partial derivatives of y with respect to $x_1, x_2, x_3 \ldots\ldots x_n$..

(ie) $\frac{\partial y}{\partial x_1} = f_1$

$\frac{\partial y}{\partial x_2} = f_2$

$$\frac{\partial y}{\partial x_3} = f_3$$

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$$\frac{\partial y}{\partial x_n} = fn$$

Step 2

Find all the second order partial derivatives of y with respect to x_1, x_2, x_3 ….x_n and they are given as follows.

$$\frac{\partial^2 y}{\partial {x_1}^2} = f_{11}$$

$$\frac{\partial^2 y}{\partial x_2 \, \partial x_1} = f_{21}$$

$$\frac{\partial^2 y}{\partial {x_2}^2} = f_{22}$$

$$\frac{\partial^2 y}{\partial x_1 \, \partial x_2} = f_{12}$$

$$\frac{\partial^2 y}{\partial {x_3}^2} = f_{33}$$

$$\frac{\partial^2 y}{\partial x_3 \, \partial x_1} = f_{31}$$

$$\frac{\partial^2 y}{\partial x_1 \, \partial x_3} = f_{13}$$ and so on

Step: 3

Construct an Hessian matrix which is formed by taking all the second order partial derivatives is given by

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \cdots\cdots\cdots f_{1n} \\ f_{21} & f_{22} & f_{23} \cdots\cdots\cdots f_{2n} \\ . & & \\ . & & \\ . & & \\ . & & \\ . & & \\ f_{n1} & f_{n2} & f_{n3} \cdots\cdots\cdots f_{nn} \end{bmatrix}$$

H is a symmetric matrix.

Step: 4

Consider the following minors of order 1, 2, 3 ……….

$$|H_1| = |f_{11}| = f_{11}$$

$$|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$|H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

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$$|H_n| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \cdots\cdots\cdots & f_{1n} \\ f_{21} & f_{22} & f_{23} \cdots\cdots\cdots & f_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ f_{n1} & f_{n2} & f_{n3} \cdots\cdots\cdots & f_{nn} \end{vmatrix}$$

Steps: 5

The necessary condition for finding the maximum or minimum

Equate the first order derivative to zero (i.e) $f_1 = f_2 = \ldots\ldots f_n = 0$ and find the value of $x_1, x_2, \ldots\ldots x_n$.

Steps: 6

Substitute the values $x_1, x_2 \ldots\ldots x_n$ in the Hessian matrix. Find the values of $|H_1|, |H_2|, |H_3| \ldots\ldots\ldots\ldots |H_n|$

If
$$|H_1| < 0$$
$$|H_2| > 0$$
$$|H_3| < 0$$
$$|H_4| > 0 \ldots\ldots\ldots\ldots \textit{ and so on.}$$

Then the function is **maximum** at $x_1, x_2 \ldots\ldots x_n$.

If $|H_1| > 0, \quad |H_2| > 0, \quad |H_3| > 0 \ldots\ldots\ldots$ then the function is **minimum** at $x_1, x_2 \ldots\ldots x_n$.

Steps: 7

Conditions	**Maximum**	**Minimum**
First	$f_1 = f_2 = f_3 = f_n = 0$	$f_1 = 0, f_2 = 0 \ldots\ldots f_n = 0$
Second	$\lvert H_1\rvert < 0$ $\lvert H_2\rvert > 0$ $\lvert H_3\rvert < 0$ ……….	$\lvert H_1\rvert > 0$ $\lvert H_2\rvert > 0$ $\lvert H_3\rvert > 0$ . . .

Note :

If the second order conditions are not satisfied then they are called **saddle point.**

Problem

Find the maxima (or) minima if any of the following function.

$$y=\frac{4}{3}x_1^3 + x_2^2 - 4x_1 + 8x_2 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(1)}$$

Solution

Step 1: The first order partial derivatives are

$$f_1 = \frac{\partial y}{\partial x_1} = 4x_1^{\,2} - 4$$

$$f_2 = \frac{\partial y}{\partial x_2} = 2x_2 + 8$$

Step 2: The second order partial derivatives are

$$f_{11} = \frac{\partial^2 y}{\partial x_1^{\,2}} = 8x_1$$

$$f_{21} = \frac{\partial^2 y}{\partial x_2\, \partial x_1} = 0$$

$$f_{22} = \frac{\partial^2 y}{\partial x_2^{\,2}} = 2$$

$$f_{12} = \frac{\partial^2 y}{\partial x_1\, \partial x_2} = 0$$

Step 3: The Hessian matrix is $H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$

$$H = \begin{bmatrix} 8x_1 & 0 \\ 0 & 2 \end{bmatrix}$$

4. Equate $f_1, f_2 = 0$

$f_1 \Rightarrow 4x_1^{2} - 4 = 0$

$x_1^2 = 1$

$x_1 = \pm 1$

$x_1 = 1, \; x_1 = -1$

$f_2 \Rightarrow 2x_2 + 8 = 0$

$2x_2 = -8$

$x_2 = -4$

The stationary points are (1,- 4) & (-1, - 4)

At the point (1, - 4) the Hessian matrix will be

$$H = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

$|H_1| = |8| > 0$

$$|H_2| = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} = 16 > 0$$

Since the determinant H_1 and H_2 are positive the function is minimum at (1, - 4).

The minimum value at $x_1 = 1$ & $x_2 = -4$ is obtained by substituting the values in (1)

$$y = \frac{4}{3}(1)^3 + (-4)^2 - 4(1) + 8(-4)$$

$$y = \frac{4}{3} + 16 - 4 - 32$$

$$y = \frac{4}{3} - 20$$

$$y = \frac{4-60}{3} = \frac{-56}{3}$$

The minimum value is $\frac{-56}{3}$

At the point (-1, - 4)

$$H = \begin{vmatrix} -8 & 0 \\ 0 & 2 \end{vmatrix}$$

$|H_1| = |-8| = -8 < 0$

$|H_2| = -16 < 0$

Both the conditions are not satisfied. Hence the point (-1, - 4) gives a saddle point.

Economic Optimum

For finding the Economic Optimum we equate the first order derivative $f_1 . f_2 .... f_n$ to the inverse ratio of the unit prices.

(ie) $f_1 = \frac{\partial y}{\partial x_1} = \frac{p_{x_1}}{p_y}$

$f_2 = \frac{\partial y}{\partial x_2} = \frac{p_{x_2}}{p_y}$ ....

$f_n = \frac{\partial y}{\partial x_n} = \frac{p_{x_n}}{p_y}$

where Px_1, Px_2,... Px_n and Py are the unit prices of x_1, x_2 ..... x_n and y. These are the first order condition.

The economic optimum & the physical optimum differ only in the first order conditions. The other procedures are the same.

Maxima & Minima of several variables under certain condition with constraints.

Consider the response function

y = f (x_1, x_2 ....x_n) subject to the constraint ϕ (x_1, x_2.....x_n) =0

The objective function is Z= f(x1, x2, …xn) + λ[ϕ(x1, x2, …xn)]

where λ is called the Lagrange's multiplies.

The partial derivatives are

$$\frac{\partial z}{\partial x_i} = fi \qquad \text{for } i = 1, 2 \ldots\ldots n.$$

$$\frac{\partial^2 z}{\partial x_i \partial x_1} = f_{ij} \qquad i, j = 1., 2 \ldots.\ n.$$

$$\frac{\partial \phi}{\partial x_i} = \phi i \qquad i = 1, 2 \ldots\ldots n.$$

Now form the Bordered Hessian Matrix as follows.

$$\text{Bordered Hessian} \qquad \overline{H} = \begin{bmatrix} 0 & \phi_1 & \phi_2 \ldots\ldots\ldots\ldots\ldots\phi \\ \phi_1 & f_{11} & f_{12} \ldots\ldots\ldots\ldots f_{1n} \\ \phi_2 & f_{21} & f_{22} \ldots\ldots\ldots\ldots f_{2n} \\ . & . & \\ . & & \\ \phi_2 & f_{n1} & f_{n2} \ldots\ldots\ldots\ldots f_{nn} \end{bmatrix}$$

[Since this extra row & column is on the border of the matrix $\begin{bmatrix} f_{11} & f_{12} \ldots\ldots\ldots f_{1n} \\ f_{21} & f_{22} \ldots\ldots\ldots f_{2n} \\ f_{n1} & f_{n2} \ldots\ldots\ldots f_{nn} \end{bmatrix}$.So we call it as Bordered Hessian matrix and it is denoted by $\overline{H}$]

Here minor as are

$$\left[\overline{H}_1\right] = \begin{vmatrix} 0 & \phi_1 \\ \phi_1 & f_{1i} \end{vmatrix}, \qquad \left|\overline{H}_2\right| = \begin{vmatrix} 0 & \phi_1 & \phi_2 \\ \phi_1 & f_{11} & f_{12} \\ \phi_2 & f_{21} & f_{22} \end{vmatrix}$$

$$|\overline{H}_3| = \begin{vmatrix} 0 & \phi_1 & \phi_2 & \phi_3 \\ \phi_1 & f_{11} & f_{12} & f_{13} \\ \phi_2 & f_{21} & f_{22} & f_{23} \\ \phi_3 & f_{31} & f_{32} & f_{33} \end{vmatrix}$$ and so on.

Problem

Conditions	Maxima	Minima
First Order	$f_1 = f_2 = f_3 = \ldots f_n = 0$	$f_1 = f_2 = f_3 \ldots\ldots f_n = 0$
Second Order	$\lvert\overline{H}_2\rvert > 0 \lvert\overline{H}_3\rvert < 0,\ \lvert\overline{H}_4\rvert > 0 \ldots.$	$\lvert\overline{\mathrm{H}}_2\rvert < 0,\ \lvert\overline{\mathrm{H}}_3\rvert < 0,\ \lvert\overline{\mathrm{H}}_4\rvert < 0 \ldots.$

Consider a consumer with a simple utility function U = f(x, y) = $4xy - y^2$. If this consumer can at most spend only Rs. 6/- on two goods x and y and if the current prices are Rs. 2/- per unit of x and Rs.1/- per unit of y. Maximize the function.